

# Modeling controllable economic growth. Proportional growth

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# Snapshot

$$y = ax_1^{\alpha_1} \dots x_n^{\alpha_n}$$

output

$$y = ax_1^{\alpha_1} \dots x_n^{\alpha_n}$$

output

$$x_1, \dots, x_n$$

factors

$$y = ax_1^{\alpha_1} \dots x_n^{\alpha_n}$$

output

$$x_1, \dots, x_n$$

factors

$$\alpha_1 + \dots + \alpha_n = 1$$

elasticities

$$y = ax_1^{\alpha_1} \dots x_n^{\alpha_n}$$

output

$$x_1, \dots, x_n$$

factors

$$\alpha_1 + \dots + \alpha_n = 1$$

elasticities

$$p_1, \dots, p_n$$

prices

$$y = ax_1^{\alpha_1} \dots x_n^{\alpha_n}$$

output

$$x_1, \dots, x_n$$

factors

$$\alpha_1 + \dots + \alpha_n = 1$$

elasticities

$$p_1, \dots, p_n$$

prices

$$C = p_1 x_1 + \dots + p_n x_n$$

cost

$$y = ax_1^{\alpha_1} \dots x_n^{\alpha_n}$$

output

$$C = p_1x_1 + \dots + p_nx_n \rightarrow \text{minimize}$$

$$p_1, \dots, p_n$$

prices

$$C = p_1x_1 + \dots + p_nx_n$$

cost

$$y = ax_1^{\alpha_1} \dots x_n^{\alpha_n}$$

$$C = p_1x_1 + \dots + p_nx_n \rightarrow \text{minimize}$$

$$\frac{x_i}{x_k} = \frac{\alpha_i}{\alpha_k} \frac{p_k}{p_i}$$

$$y = ax_1^{\alpha_1} \dots x_n^{\alpha_n}$$

$$C = p_1 x_1 + \dots + p_n x_n \rightarrow \text{minimize}$$

$$\frac{x_i}{x_k} = \frac{\alpha_i}{\alpha_k} \frac{p_k}{p_i}$$

$$x_j = C \frac{\alpha_j}{p_j}$$

$$y = ax_1^{\alpha_1} \dots x_n^{\alpha_n}$$

$$C = p_1 x_1 + \dots + p_n x_n \rightarrow \text{minimize}$$

$$\frac{x_i}{x_k} = \frac{\alpha_i}{\alpha_k} \frac{p_k}{p_i} \quad | \quad x_j = C \frac{\alpha_j}{p_j}$$

$$x_j = \frac{y^{\alpha_j}}{a} \left( \frac{p_1}{\alpha_1} \right)^{\alpha_1} \dots \left( \frac{p_n}{\alpha_n} \right)^{\alpha_n}$$

$$x_j = C \frac{\alpha_j}{p_j}$$

$$x_j = \frac{y \alpha_j}{a p_j} \left( \frac{P_1}{\alpha_1} \right)^{\alpha_1} \dots \left( \frac{P_n}{\alpha_n} \right)^{\alpha_n}$$

$$y = A C$$

$$A = a \left( \frac{\alpha_1}{p_1} \right)^{\alpha_1} \dots \left( \frac{\alpha_n}{p_n} \right)^{\alpha_n}$$

$$x_j = C \frac{\alpha_j}{p_j}$$

$$x_j = \frac{y \alpha_j}{a p_j} \left( \frac{p_1}{\alpha_1} \right)^{\alpha_1} \dots \left( \frac{p_n}{\alpha_n} \right)^{\alpha_n}$$

$$y = A C$$

$$A = a \left( \frac{\alpha_1}{p_1} \right)^{\alpha_1} \dots \left( \frac{\alpha_n}{p_n} \right)^{\alpha_n}$$

$$x_j = C \frac{\alpha_j}{p_j}$$

$$x_j = \frac{y \alpha_j}{a p_j} \left( \frac{p_1}{\alpha_1} \right)^{\alpha_1}$$

$$x_j = \frac{y \alpha_j}{A p_j}$$

$$y = A C$$

$$A = a \left( \frac{\alpha_1}{p_1} \right)^{\alpha_1} \dots \left( \frac{\alpha_n}{p_n} \right)^{\alpha_n}$$

$$p_j x_j = \alpha_j C$$

$$x_j = C \frac{\alpha_j}{p_j}$$

$$x_j = y \frac{\alpha_j}{A p_j}$$

$$y = A C$$

$$p_j x_j = \alpha_j C$$

$$y = A C$$

$$p_j x_j = \alpha_j C$$

$$y = A C$$

$$y = A C$$

$$p_j x_j = \alpha_j C$$

The diagram illustrates the decomposition of a matrix equation. At the top, a large gray rectangle contains the equation  $y = A C$ , where the  $C$  term is highlighted with a brown border. An orange arrow points from this term down to a smaller gray rectangle below. This smaller rectangle contains the equation  $p_j x_j = \alpha_j C$ , where the  $C$  term is also highlighted with a brown border. From this second  $C$  term, two orange arrows point downwards to a white horizontal bar at the bottom. On the left side of this bar is the equation  $y = A C$ , with the  $C$  term highlighted by a green border. On the right side is the equation  $p_j x_j = \alpha_j C$ , with the  $C$  term highlighted by an orange border.

$$y = A C$$

$$y = A C$$

$$p_j x_j = \alpha_j C$$

$$y = A\boxed{C} \quad p_jx_j = \alpha_j\boxed{C}$$

# Dynamics

$$y = A \boxed{C} \quad p_j x_j = \alpha_j \boxed{C}$$

$$y = I + Z$$

output

$$y = A[C] p_j x_j = \alpha_j [C]$$

$$y = I + Z$$

output

$$I = sy \quad (s \leq 1)$$

investment

$$y = A[C] \quad p_j x_j = \alpha_j [C]$$

$$y = I + Z$$

output

$$I = sy \quad (s \leq 1)$$

investment

$$Z = (1 - s)y$$

consumption

$$y = A[C] \quad p_j x_j = \alpha_j [C]$$

$$y = I + Z$$

output

$$I = sy \quad (s \leq 1)$$

investment

$$Z = (1 - s)y$$

consumption

$$I_j = s_j y$$

investment in  $x_j$

$$y = A[C] \quad p_j x_j = \alpha_j [C]$$

$$y = I + Z$$

output

$$I = sy \quad (s \leq 1)$$

investment

$$Z = (1 - s)y$$

consumption

$$I_j = s_j y$$

investment in  $x_j$

$$s_1 + \dots + s_n = s$$

$$y = A C$$

$$p_j x_j = \alpha_j C$$

$$y = I + Z$$

output

$$I = sy \quad (s \leq 1)$$

investment

$$Z = (1 - s)y$$

consumption

$$I_j = s_j y$$

investment in  $x_j$

$$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$$

inflow in  $x_j$

$$s_1 + \dots + s_n = s$$

$$y = A C$$

$$p_j x_j = \alpha_j C$$

$$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$$

inflow in  $x_j$

$$s_1 + \dots + s_n = s$$

$$y = A C$$

$$p_j x_j = \alpha_j C$$

$$\dot{x}_j = \xi_j - \delta_j x_j$$

growth rate in  $x_j$

$$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$$

inflow in  $x_j$

$$s_1 + \dots + s_n = s$$

$$y = A C$$

$$p_j x_j = \alpha_j C$$

$$\dot{x}_j = s_j \frac{y}{p_j} - \delta_j x_j$$

$$\dot{x}_j = \xi_j - \delta_j x_j$$

growth rate in  $x_j$

$$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$$

inflow in  $x_j$

$$s_1 + \dots + s_n = s$$

$$y = A C$$

$$p_j x_j = \alpha_j C$$

$$\dot{x}_j = s_j \frac{y}{p_j} - \delta_j x_j = s_j \frac{AC}{p_j} -$$

$$\dot{x}_j = \xi_j - \delta_j x_j$$

growth rate in  $x_j$

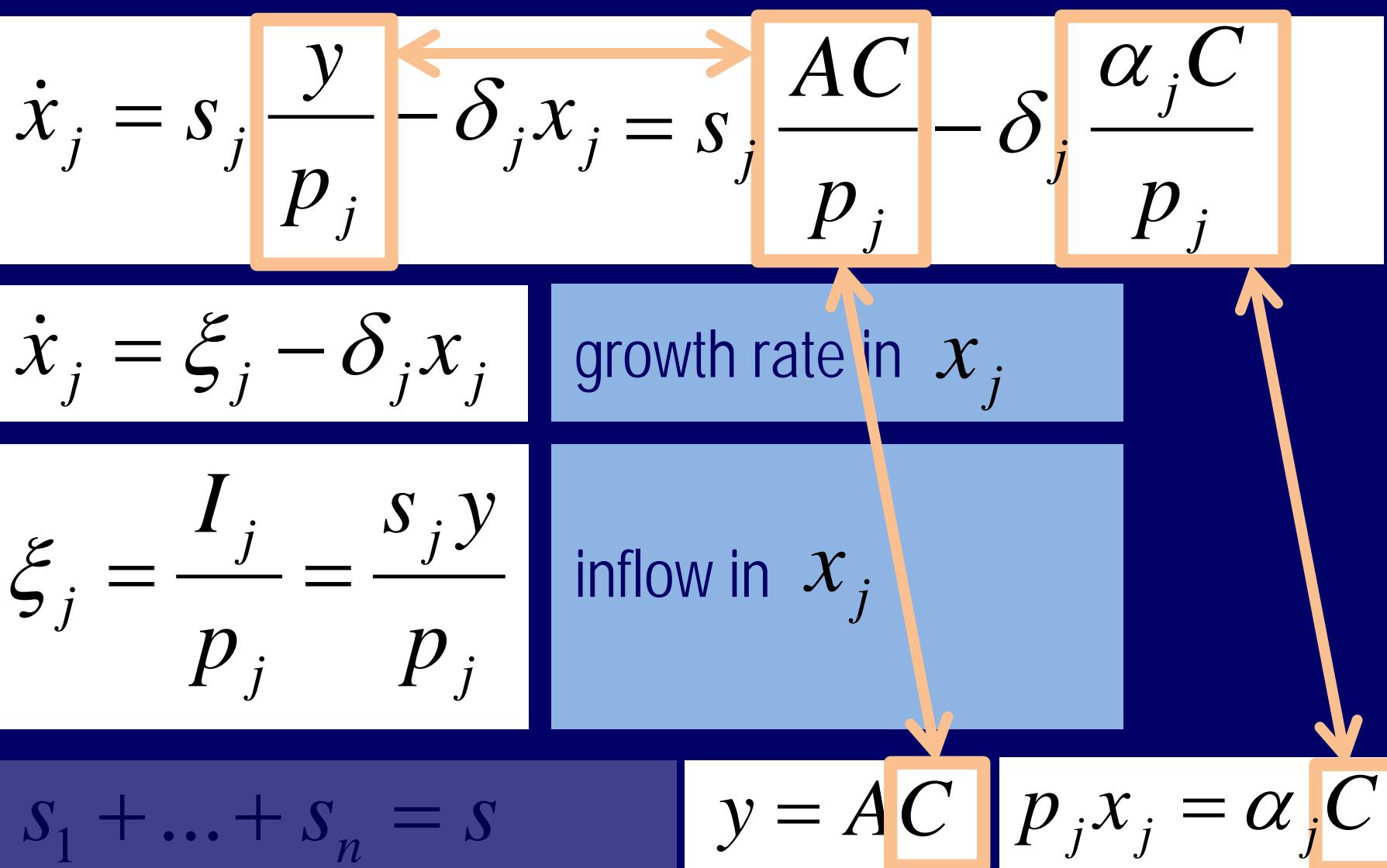
$$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$$

inflow in  $x_j$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$



$$\dot{x}_j = s_j \frac{y}{p_j} - \delta_j x_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y=AC$$

$$p_jx_j=\alpha_jC$$

$$\dot{x}_j = s_j \frac{y}{p_j} - \delta_j x_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$



$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y=AC$$

$$p_jx_j=\alpha_jC$$

$$C = p_1x_1 + \dots + p_nx_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_jx_j = \alpha_j C$$

$$\dot{C}=\dot{p}_1x_1+...+\dot{p}_nx_n+p_1\dot{x}_1+...+p_n\dot{x}_n$$

$$C=p_1x_1+...+p_nx_n$$

$$\dot{x}_j=s_j\,\frac{AC}{p_j}-\delta_j\,\frac{\alpha_j C}{p_j}$$

$$s_1+...+s_n=s$$

$$y=AC$$

$$p_jx_j=\alpha_jC$$

$$\dot{C} = \dot{p}_1 x_1 + \dots + \dot{p}_n x_n + p_1 \dot{x}_1 + \dots + p_n \dot{x}_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = \dot{p}_1 x_1 + \dots + \dot{p}_n x_n + p_1 \dot{x}_1 + \dots + p_n \dot{x}_n$$

The diagram illustrates the decomposition of the total derivative of  $C$  into partial derivatives of  $p_i$  and  $x_j$ . The top equation shows the total derivative  $\dot{C}$  as a sum of terms involving  $\dot{p}_i x_i$  and  $p_i \dot{x}_i$ . The terms  $\dot{p}_i x_i$  are highlighted with orange boxes, and the terms  $p_i \dot{x}_i$  are highlighted with green boxes. Arrows point from the terms  $\dot{p}_1 x_1$  and  $\dot{p}_n x_n$  to the term  $\dot{x}_j$ , which is also highlighted with a green box. This indicates that the total derivative  $\dot{C}$  can be expressed as a sum of terms where each term is a product of a partial derivative of  $C$  with respect to  $p_i$  and the corresponding  $x_i$ , plus a sum of terms where each term is a product of a partial derivative of  $C$  with respect to  $x_j$  and the corresponding  $p_i$ .

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = \dot{p}_1 \boxed{x_1} + \dots + \dot{p}_n \boxed{x_n} + p_1 \boxed{\dot{x}_1} + \dots + p_n \boxed{\dot{x}_n}$$

$$\dot{C} = \alpha_1 \frac{\dot{p}_1}{p_1} C + \dots + \alpha_n \frac{\dot{p}_n}{p_n} C +$$

$$s_1 A C + \dots + s_n A C - \alpha_1 \delta_1 C - \dots - \alpha_n \delta_n C$$

$$\boxed{\dot{x}_j} = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j \boxed{x_j} = \alpha_j C$$

$$\dot{C} = \dot{p}_1 \boxed{x_1} + \dots + \dot{p}_n \boxed{x_n} + p_1 \boxed{\dot{x}_1} + \dots + p_n \boxed{\dot{x}_n}$$

$$\dot{C} = \alpha_1 \frac{\dot{p}_1}{p_1} C + \dots + \alpha_n \frac{\dot{p}_n}{p_n} C +$$

$$s_1 A C + \dots + s_n A C - \alpha_1 \delta_1 C - \dots - \alpha_n \delta_n C$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \dots + \alpha_n \frac{\dot{p}_n}{p_n}$$
$$\delta = \alpha_1 \delta_1 + \dots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \dots + \alpha_n \frac{\dot{p}_n}{p_n}$$
$$\delta = \alpha_1 \delta_1 + \dots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \dots + \alpha_n \frac{\dot{p}_n}{p_n}$$
$$\delta = \alpha_1 \delta_1 + \dots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$\dot{x}_j = S_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$S_1 + \dots + S_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \dots + \alpha_n \frac{\dot{p}_n}{p_n}$$
$$\delta = \alpha_1 \delta_1 + \dots + \alpha_n \delta_n$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \dots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \dots + \alpha_n \delta_n$$

$$\dot{x}_j = S_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$S_1 + \dots + S_n = S$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \dots + \alpha_n \frac{\dot{p}_n}{p_n}$$
$$\delta = \alpha_1 \delta_1 + \dots + \alpha_n \delta_n$$

$$\dot{x}_j = S_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$S_1 + \dots + S_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \dots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \dots + \alpha_n \delta_n$$

$$\dot{x}_j = S_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$S_1 + \dots + S_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$s_j = \alpha_j s + \frac{\alpha_j}{A} \left[ \left( r - \frac{p_j}{p} \right) - (\delta - \delta_j) \right] + \frac{\dot{\alpha}_j}{A}$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \dots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \dots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$s_j = \alpha_j s + \frac{\alpha_j}{A} \left[ \left( r - \frac{p_j}{p} \right) - (\delta - \delta_j) \right] + \frac{\dot{\alpha}_j}{A}$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \dots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \dots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$s_j = \alpha_j s + \frac{\alpha_j}{A} \left[ \left( r - \frac{\dot{p}_j}{p_j} \right) - (\delta - \delta_j) \right] + \frac{\dot{\alpha}_j}{A}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \dots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \dots + \alpha_n \delta_n$$

$$\dot{x}_j = \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$A = a \left( \frac{\alpha_1}{p_1} \right)^{\alpha_1} \dots \left( \frac{\alpha_n}{p_n} \right)^{\alpha_n}$$

$$s_1 + \dots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

# Optimal control

$$\dot{C} = C(sA + r - \delta)$$

$$y=AC$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0)=C_0$$

$$y=AC$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1 - s)y$$

consumption flow

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1 - s)y$$

consumption flow

$$J = \int_0^\infty e^{-\rho t} \log c dt$$

utility

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1 - s)y$$

consumption flow

$$J = \int_0^\infty e^{-\rho t} \log c dt$$

utility

$$\log c = \log(1 - s) + \log y = \log(1 - s) + \log C + \log A$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1-s)y$$

$$J = \int_0^\infty e^{-\rho t} \log c dt$$

$$\log c = \log(1-s) - \log y = \log(1-s) + \log C + \log A$$

consumption flow

utility

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1 - s)y$$

consumption flow

$$J = \int_0^\infty e^{-\rho t} \log c dt$$

utility

$$\log c = \log(1 - s) + \log y = \log(1 - s) + \log C + \log A$$

$$J = \int_0^\infty e^{-\rho t} [\log(1 - s) + \log C] dt + K$$

$$K = \int_0^\infty e^{-\rho t} \log A dt$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1 - s)y$$

consumption flow

$$J = \int_0^\infty e^{-\rho t} \log c dt$$

utility

$$\log c = \log(1 - s) + \log y = \log(1 - s) + \log C + \log A$$

$$J = \int_0^\infty e^{-\rho t} [\log(1 - s) + \log C] dt + K$$

$$K = \int_0^\infty e^{-\rho t} \log A dt$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0)=C_0$$

$$\varepsilon\leq s\leq 1$$

$$J=\int_0^{\infty}e^{-\rho t}[\log(1-s)+\log C]dt+K\rightarrow\max$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$\varepsilon \leq s \leq 1$$

$$J = \int_0^\infty e^{-\rho t} [\log(1-s) + \log C] dt + K \rightarrow \max$$

$$s = \begin{cases} 1 - \frac{\rho}{A}, & A(1-\varepsilon) \geq \rho \\ \varepsilon, & A(1-\varepsilon) < \rho \end{cases}$$

optimal investment

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$\varepsilon \leq s \leq 1$$

$$J = \int_0^\infty e^{-\rho t} [\log(1-s) + \log C] dt + K \rightarrow \max$$

$$s = \begin{cases} 1 - \frac{\rho}{A}, & A(1-\varepsilon) \geq \rho \\ \varepsilon, & A(1-\varepsilon) < \rho \end{cases}$$

optimal investment

$$s_j = \alpha_j s + \frac{\alpha_j}{A} \left[ \left( r - \frac{\dot{p}_j}{p_j} \right) - (\delta - \delta_j) \right] + \frac{\dot{\alpha}_j}{A}$$

Thank you