Modeling controllable economic growth. Proportional growth

Arkady Kryazhimskiy
International Institute for Applied Systems Analysis (Laxenburg, Austria)
Steklov institute of Mathematics, RAS (Moscow, Russia)

Keio University, 4-8 February, 2013

Snapshot

$$y = a x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

output

$$y = a x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

output

$$x_1, \ldots, x_n$$

factors

$y = a x_1^{\alpha_1} \ldots x_n^{\alpha_n}$	output
$x_1, \ldots, x_n$	factors
$\alpha_1 + \ldots + \alpha_n = 1$	elasticities

$y = a x_1^{\alpha_1} \dots x_n^{\alpha_n}$	output
$x_1, \dots, x_n$	factors
$\alpha_1 + \dots + \alpha_n = 1$	elasticities
$p_1, \dots, p_n$	prices

$y = a x_1^{\alpha_1} \dots x_n^{\alpha_n}$	output
$x_1, \dots, x_n$	factors
$\alpha_1 + \dots + \alpha_n = 1$	elasticities
$p_1, \dots, p_n$	prices
$C = p_1 x_1 + \dots + p_n x_n$	cost

$$y = a x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$ output

$$C = p_1 x_1 + \ldots + p_n x_n \rightarrow$$ minimize

$$p_1, \ldots, p_n$$ prices

$$C = p_1 x_1 + \ldots + p_n x_n$$ cost

$$y = a x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

$$C = p_1 x_1 + \ldots + p_n x_n \rightarrow \text{minimize}$$

$$\frac{x_i}{x_k} = \frac{\alpha_i}{\alpha_k} \frac{p_k}{p_i}$$

$$y = a x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

$$C = p_1 x_1 + \dots + p_n x_n \rightarrow \text{minimize}$$

$$\frac{x_i}{x_k} = \frac{\alpha_i}{\alpha_k} \frac{p_k}{p_i}$$

$$x_j = C \frac{\alpha_j}{p_j}$$

$$y = a x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

$$C = p_1 x_1 + \dots + p_n x_n \to \text{minimize}$$

$$\frac{x_i}{x_k} = \frac{\alpha_i}{\alpha_k} \frac{p_k}{p_i}$$

$$x_j = C \frac{\alpha_j}{p_j}$$

$$x_j = \frac{y}{a} \frac{\alpha_j}{p_j} \left(\frac{p_1}{\alpha_1} \right)^{\alpha_1} \dots \left(\frac{p_n}{\alpha_n} \right)^{\alpha_n}$$

$$x_j = C\frac{\alpha_j}{p_j}$$

$$x_j = \frac{y}{a}\frac{\alpha_j}{p_j}\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1}\cdots\left(\frac{p_n}{\alpha_n}\right)^{\alpha_n}$$

$$y = AC$$

$$A = a\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} \cdots \left(\frac{\alpha_n}{p_n}\right)^{\alpha_n}$$

$$x_j = C\frac{\alpha_j}{p_j}$$

$$x_j = \frac{y}{a}\frac{\alpha_j}{p_j}\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1} \cdots \left(\frac{p_n}{\alpha_n}\right)^{\alpha_n}$$

$$y = AC$$

$$A = a\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} \cdots \left(\frac{\alpha_n}{p_n}\right)^{\alpha_n}$$

$$x_j = C\frac{\alpha_j}{p_j}$$

$$x_j = y\frac{\alpha_j}{Ap_j}$$

$$x_j = \frac{y}{a}\frac{\alpha_j}{p_j}\left(\frac{p_1}{\alpha_1}\right)^{\alpha} \cdots$$

$$y = AC$$

$$A = a\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} \cdots \left(\frac{\alpha_n}{p_n}\right)^{\alpha_n}$$

$$p_j x_j = \alpha_j C$$

$$x_j = C\frac{\alpha_j}{p_j}$$

$$x_j = y\frac{\alpha_j}{Ap_j}$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$y = AC$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$y = AC \qquad p_j x_j = \alpha_j C$$

$y = AC$ $p_j x_j = \alpha_j C$

Dynamics

$y = AC$ $p_j x_j = \alpha_j C$

$$y = I + Z$$

output

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$y = I + Z$	output
$I = sy \quad (s \leq 1)$	investment

$$y = AC \qquad p_j x_j = \alpha_j C$$

$y = I + Z$	output
$I = sy \quad (s \leq 1)$	investment
$Z = (1 - s)y$	consumption

$y = AC$ $\quad$ $p_j x_j = \alpha_j C$

$y = I + Z$	output
$I = sy \quad (s \leq 1)$	investment
$Z = (1 - s)y$	consumption
$I_j = s_j y$	investment in x_j

$y = AC$ $\quad$ $p_j x_j = \alpha_j C$

$y = I + Z$	output
$I = sy \quad (s \le 1)$	investment
$Z = (1 - s)y$	consumption
$I_j = s_j y$	investment in x_j

$s_1 + \ldots + s_n = s$

$y = AC$

$p_j x_j = \alpha_j C$

$y = I + Z$	output
$I = sy \quad (s \le 1)$	investment
$Z = (1 - s)y$	consumption
$I_j = s_j y$	investment in x_j
$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$	inflow in x_j

$s_1 + \ldots + s_n = s$

$y = AC$

$p_j x_j = \alpha_j C$

$$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$$

inflow in x_j

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{x}_j = \xi_j - \delta_j x_j$$

growth rate in x_j

$$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$$

inflow in x_j

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{x}_j = s_j \frac{y}{p_j} - \delta_j x_j$$

$$\dot{x}_j = \xi_j - \delta_j x_j$$

growth rate in x_j

$$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$$

inflow in x_j

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{x}_j = s_j \frac{y}{p_j} - \delta_j x_j = s_j \frac{AC}{p_j} -$$

$$\dot{x}_j = \xi_j - \delta_j x_j$$

growth rate in x_j

$$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$$

inflow in x_j

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{x}_j = s_j \frac{y}{p_j} - \delta_j x_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$\dot{x}_j = \xi_j - \delta_j x_j$$

growth rate in x_j

$$\xi_j = \frac{I_j}{p_j} = \frac{s_j y}{p_j}$$

inflow in x_j

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{x}_j = s_j \frac{y}{p_j} - \delta_j x_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s \qquad y = AC \qquad p_j x_j = \alpha_j C$$

$$\dot{x}_j = s_j \frac{y}{p_j} - \delta_j x_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$C = p_1 x_1 + \ldots + p_n x_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = \dot{p}_1 x_1 + ... + \dot{p}_n x_n + p_1 \dot{x}_1 + ... + p_n \dot{x}_n$$

$$C = p_1 x_1 + ... + p_n x_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + ... + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = \dot{p}_1 x_1 + \ldots + \dot{p}_n x_n + p_1 \dot{x}_1 + \ldots + p_n \dot{x}_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = \dot{p}_1 x_1 + \ldots + \dot{p}_n x_n + p_1 \dot{x}_1 + \ldots + p_n \dot{x}_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = \dot{p}_1 x_1 + \ldots + \dot{p}_n x_n + p_1 \dot{x}_1 + \ldots + p_n \dot{x}_n$$

$$\dot{C} = \alpha_1 \frac{\dot{p}_1}{p_1} C + \ldots + \alpha_n \frac{\dot{p}_n}{p_n} C + s_1 AC + \ldots + s_n AC - \alpha_1 \delta_1 C - \ldots - \alpha_n \delta_n C$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = \dot{p}_1 x_1 + \ldots + \dot{p}_n x_n + p_1 \dot{x}_1 + \ldots + p_n \dot{x}_n$$

$$\dot{C} = \alpha_1 \frac{\dot{p}_1}{p_1} C + \ldots + \alpha_n \frac{\dot{p}_n}{p_n} C +$$

$$s_1 AC + \ldots + s_n AC - \alpha_1 \delta_1 C - \ldots - \alpha_n \delta_n C$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \ldots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \ldots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \ldots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \ldots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \ldots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \ldots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \ldots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \ldots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \ldots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \ldots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \ldots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \ldots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \ldots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \ldots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$s_j = \alpha_j s + \frac{\alpha_j}{A}\left[\left(r - \frac{p_j}{p}\right) - (\delta - \delta_j)\right] + \frac{\dot{\alpha}_j}{A}$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \ldots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \ldots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$s_j = \alpha_j s + \frac{\alpha_j}{A}\left[\left(r - \frac{p_j}{p}\right) - (\delta - \delta_j)\right] + \frac{\dot{\alpha}_j}{A}$$

$$p_j \dot{x}_j + \dot{p}_j x_j = \dot{\alpha}_j C + \alpha_j \dot{C}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \ldots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \ldots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

$$s_j = \alpha_j s + \frac{\alpha_j}{A}\left[\left(r - \frac{\dot{p}_j}{p_j}\right) - (\delta - \delta_j)\right] + \frac{\dot{\alpha}_j}{A}$$

$$\dot{C} = C(sA + r - \delta)$$

$$r = \alpha_1 \frac{\dot{p}_1}{p_1} + \ldots + \alpha_n \frac{\dot{p}_n}{p_n}$$

$$\delta = \alpha_1 \delta_1 + \ldots + \alpha_n \delta_n$$

$$\dot{x}_j = s_j \frac{AC}{p_j} - \delta_j \frac{\alpha_j C}{p_j}$$

$$A = a\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} \ldots \left(\frac{\alpha_n}{p_n}\right)^{\alpha_n}$$

$$s_1 + \ldots + s_n = s$$

$$y = AC$$

$$p_j x_j = \alpha_j C$$

Optimal control

$$\dot{C} = C(sA + r - \delta)$$

$$y = AC$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1 - s)y$$

consumption flow

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1 - s)y$$

consumption flow

$$J = \int_0^\infty e^{-\rho t} \log c \, dt$$

utility

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1 - s)y$$

consumption flow

$$J = \int_0^\infty e^{-\rho t} \log c \, dt$$

utility

$$\log c = \log(1-s) + \log y = \log(1-s) + \log C + \log A$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1-s)y$$

consumption flow

$$J = \int_0^\infty e^{-\rho t} \log c \, dt$$

utility

$$\log c = \log(1-s) + \log y = \log(1-s) + \log C + \log A$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1 - s)y$$

consumption flow

$$J = \int_0^\infty e^{-\rho t} \log c \, dt$$

utility

$$\log c = \log(1 - s) + \log y = \log(1 - s) + \log C + \log A$$

$$J = \int_0^\infty e^{-\rho t} [\log(1 - s) + \log C] dt + K$$

$$K = \int_0^\infty e^{-\rho t} \log A \, dt$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$y = AC$$

$$c = (1 - s)y$$

consumption flow

$$J = \int_0^\infty e^{-\rho t} \log c\, dt$$

utility

$$\log c = \log(1-s) + \log y = \log(1-s) + \log C + \log A$$

$$J = \int_0^\infty e^{-\rho t} [\log(1-s) + \log C] dt + K$$

$$K = \int_0^\infty e^{-\rho t} \log A\, dt$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$\varepsilon \le s \le 1$$

$$J = \int_0^\infty e^{-\rho t}[\log(1 - s) + \log C]dt + K \to \max$$

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$\varepsilon \le s \le 1$$

$$J = \int_0^\infty e^{-\rho t}[\log(1-s) + \log C]dt + K \to \max$$

$$s = \begin{cases} 1 - \dfrac{\rho}{A}, & A(1-\varepsilon) \ge \rho \\ \varepsilon, & A(1-\varepsilon) < \rho \end{cases}$$

optimal investment

$$\dot{C} = C(sA + r - \delta)$$

$$C(0) = C_0$$

$$\varepsilon \le s \le 1$$

$$J = \int_0^\infty e^{-\rho t}[\log(1-s) + \log C]dt + K \to \max$$

$$s = \begin{cases} 1 - \dfrac{\rho}{A}, & A(1-\varepsilon) \ge \rho \\ \varepsilon, & A(1-\varepsilon) < \rho \end{cases}$$

optimal investment

$$s_j = \alpha_j s + \frac{\alpha_j}{A}\left[\left(r - \frac{\dot{p}_j}{p_j}\right) - (\delta - \delta_j)\right] + \frac{\dot{\alpha}_j}{A}$$

Thank you