Modeling controllable economic growth. Reasoning behind infinite-horizon maximim principle

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State equation

$$\dot{x}(t) = f\left(\underbrace{x(t)}_{\text{state}}, \underbrace{u(t)}_{\text{control}}\right)$$

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Constraints

$$x(0) = x_0$$

$$u(t) \in U$$

State equation

$$\dot{x}(t) = f\left(x(t), u(t)\right)$$

state control

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$$x(0) = x_0$$

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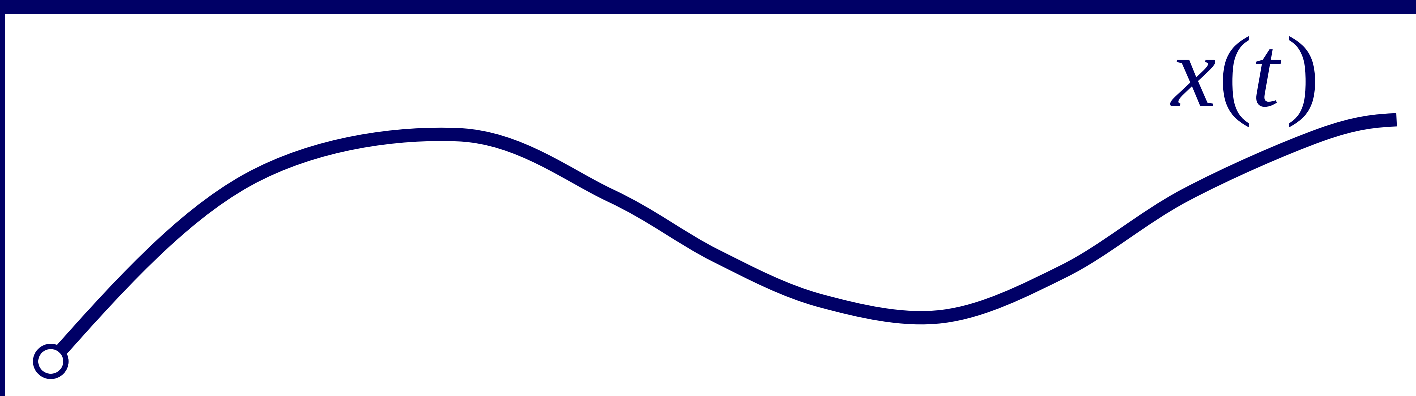

State equation

$$\dot{x}(t) = f(x(t), u(t))$$

$\underbrace{x(t)}_{\text{state}}$ $\underbrace{u(t)}_{\text{control}}$

Constraints

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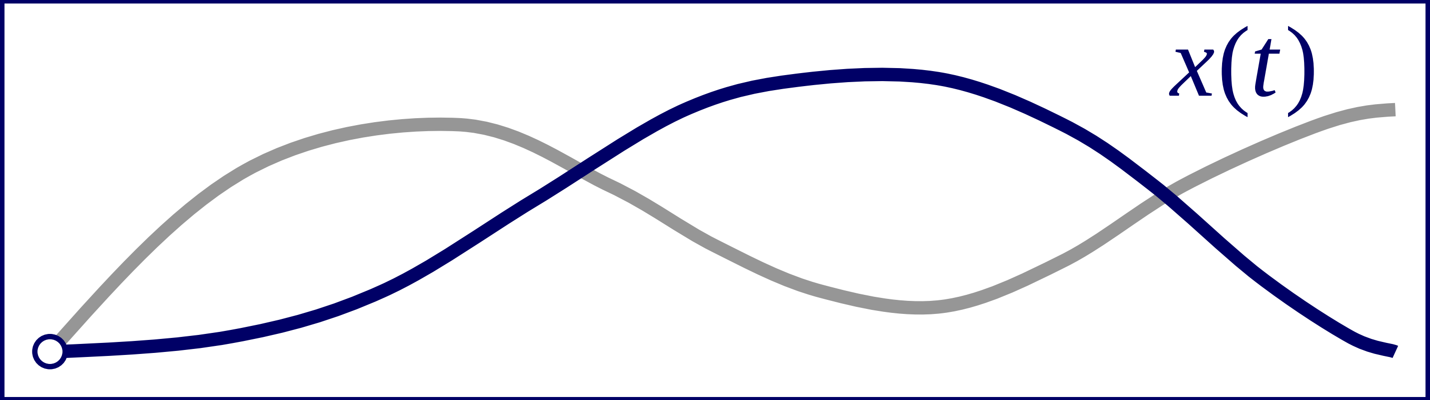

State equation

$$\dot{x}(t) = f\,(x(t), u(t))$$

state control

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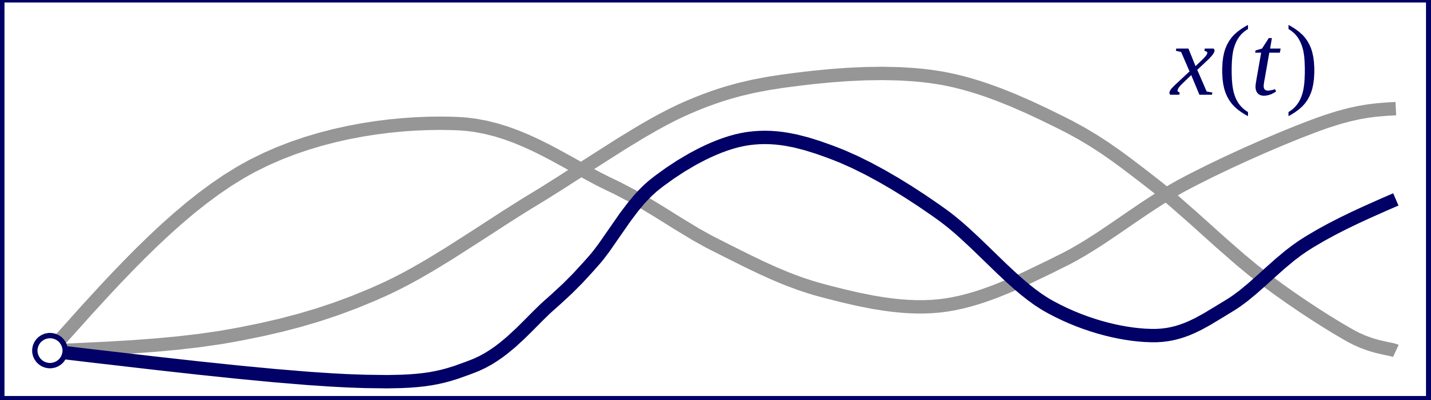

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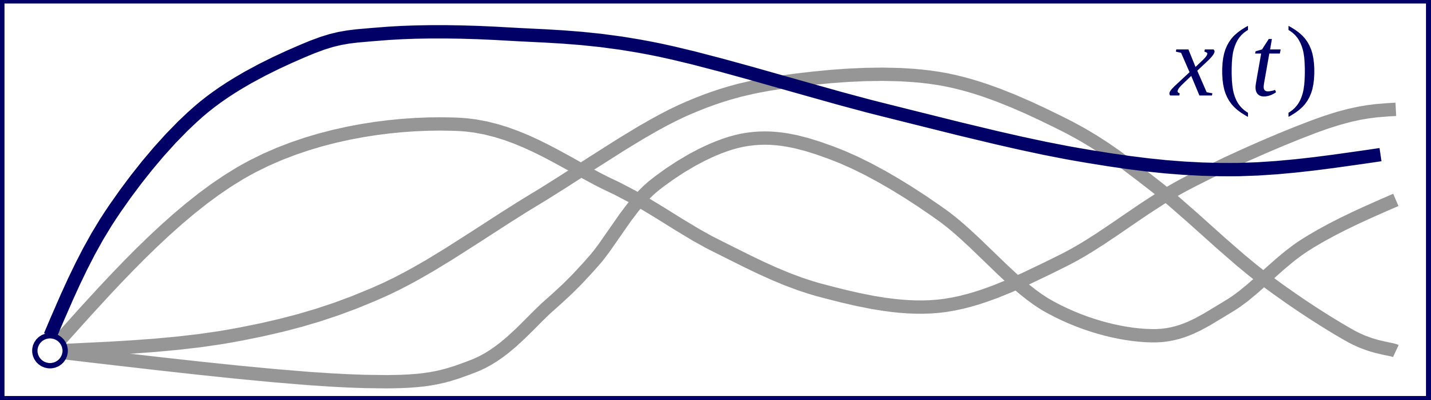

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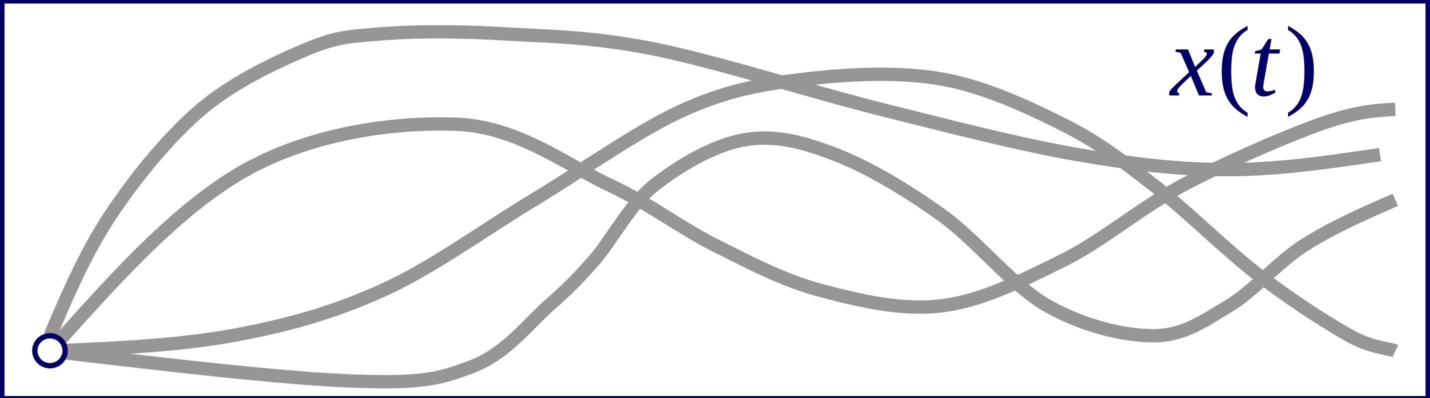


State equation

$$\dot{x}(t) = f(\underbrace{x(t)}_{\text{state}}, \underbrace{u(t)}_{\text{control}})$$

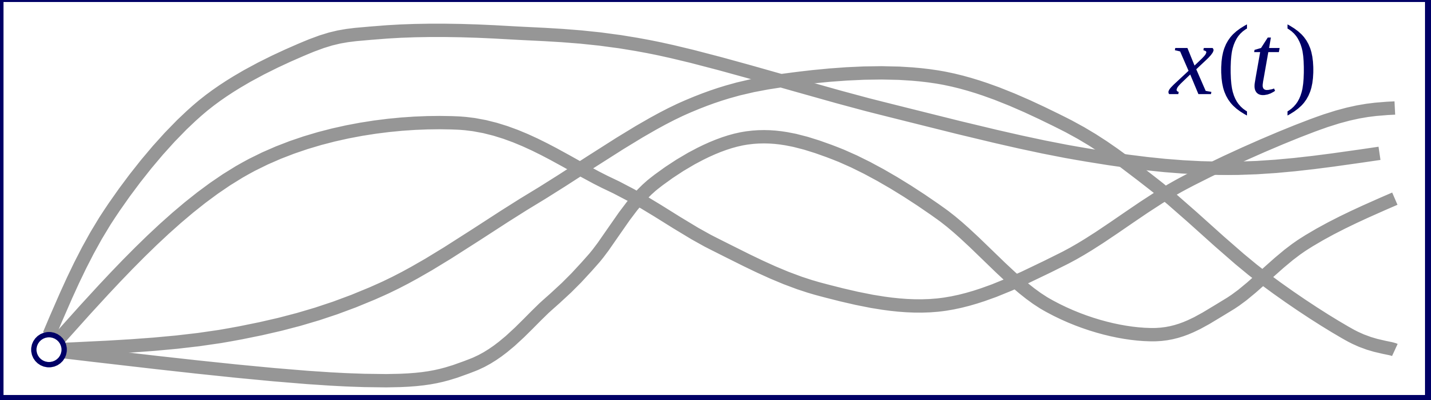
Constraints

$$x(0) = x_0$$

$$u(t) \in U$$

Utility

$$J = \int_0^\infty e^{-\rho t} \underbrace{g(x(t), u(t))}_{\text{utility flow}} dt$$



State equation

$$\dot{x}(t) = f(x(t), u(t))$$

state control

Constraints

$$x(0) = x_0$$

$$u(t) \in U$$

Utility

$$J = \int_0^\infty e^{-\rho t} g(x(t), u(t)) dt \rightarrow \textbf{MAX}$$

utility flow

Capital flow

$$\dot{C} = \psi \dot{x} + e^{-\rho t} g$$

shadow price

Capital flow

$$\dot{C} = \psi \dot{x} + e^{-\rho t} g$$

shadow price

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g$$

shadow price

Capital flow

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shadow price

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g$$

shadow price

Maximum flow

$$u(t) : \dot{C} \rightarrow \max$$

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g = H$$

shadow price Hamiltonian

Maximum flow

$$u(t) : \dot{C} \to \max$$

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g = H$$

shadow price Hamiltonian

Maximum flow

$$u(t): H \longrightarrow \max$$

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g = H$$

shadow price Hamiltonian

Maximum flow

$$u(t) : H \to \max$$

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g = H$$

shadow price Hamiltonian

Maximum flow

$$u(t) : H \to \max$$

Full capital

$$C(\infty) = J$$

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g = H$$

shadow price Hamiltonian

Maximum flow

$$u(t) : H \rightarrow \max$$

Full capital

$$C(\infty) = J$$

Future capital

$$J - C$$

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g = H$$

shadow price Hamiltonian

Maximum flow

$$u(t) : H \to \max$$

Full capital

$$C(\infty) = J$$

Future capital

$$J - C$$

Optimal flow

$$\dot{C} = \rho (J - C)$$

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g = H$$

shadow price Hamiltonian

Maximum flow

$$u(t) : H \to \max$$

Full capital

$$C(\infty) = J$$

Future capital

$$J - C$$

Optimal flow

$$\dot{C} = \rho(J - C) = \rho \int_t^{\infty} e^{-\rho s} g\, ds$$

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g = H$$

shadow price Hamiltonian

Maximum flow

$$u(t) : H \rightarrow \max$$

Full capital

$$C(\infty) = J$$

Future capital

$$J - C$$

Optimal flow

$$\dot{C} = \rho(J - C) = \rho \int_t^\infty e^{-\rho s} g \, ds$$

Capital flow

$$\dot{C} = \psi f + e^{-\rho t} g = H$$

shadow price Hamiltonian

Maximum flow

$$u(t) : H \to \max$$

Full capital

$$C(\infty) = J$$

Future capital

$$J - C$$

Optimal flow

$$H = \rho \int_t^\infty e^{-\rho s} g\, ds$$

Optimal flow

$$\dot{C} = \rho (J - C) = \rho \int_t^\infty e^{-\rho s} g\, ds$$

Maximum flow

$$u(t): H \to \max$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g ds$$

Maximum flow

$$u(t) : H \to \max$$

Optimal flow

$$H = \rho \int_{t}^{\infty} e^{-\rho s} g ds$$

Maximum flow

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Maximum flow

$$u(t): H \to \max$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g \, ds$$

$$H = \psi f + e^{-\rho t} g \to 0 \quad (t \to \infty)$$

Maximum flow

$$u(t): H \to \max$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g\, ds$$

$$H = \psi f + e^{-\rho t} g \to 0 \quad (t \to \infty)$$

$$\psi \to 0 \quad (t \to \infty)$$

Maximum flow

$$u(t): H \rightarrow \max$$

Optimal flow

$$H = \rho \int_{t}^{\infty} e^{-\rho s} g ds$$

Transversality

$$\psi \rightarrow 0 \quad (t \rightarrow \infty)$$

Maximum flow

$$u(t): H \to \max$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g ds$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Maximum flow

$$u(t): H \rightarrow \max$$

Optimal flow

$$H = \rho \int_t^\infty e^{-\rho s} g ds$$

Transversality

$$\psi \rightarrow 0 \quad (t \rightarrow \infty)$$

$$\dot{H} = \dot{\psi} f + \psi \frac{\partial f}{\partial x}\dot{x} + e^{-\rho t}\frac{\partial g}{\partial x}\dot{x} - \rho e^{-\rho t} g$$

$$= -\rho e^{-\rho t} g$$

n flow

$$\to \max$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g ds$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

$$\dot{H} = \dot{\psi} f + \psi \frac{\partial f}{\partial x} \dot{x} + e^{-\rho t} \frac{\partial g}{\partial x} \dot{x} - \rho e^{-\rho t} g$$

$$\dot{\psi} = \dot{x}$$

$$= -\rho e^{-\rho t} g$$

...m flow

$$\ldots \to \max$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g ds$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

n flow

$$\to \max$$

$$\dot{H} = \dot{\psi} f + \psi \frac{\partial f}{\partial x}\dot{x} + e^{-\rho t}\frac{\partial g}{\partial x}\dot{x} - \cancel{\rho e^{-\rho t} g}$$

$$\dot{\psi} = \dot{x}$$

$$= -\cancel{\rho e^{-\rho t} g}$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g\, ds$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

$$\dot{H} = \dot{\psi} f + \psi \frac{\partial f}{\partial x} \dot{x} + e^{-\rho t} \frac{\partial g}{\partial x} \dot{x} - \cancel{\rho e^{-\rho t}} g$$

$$f = \dot{x}$$

$$= -\cancel{\rho e^{-\rho t}} g$$

n flow

$$\rightarrow \max$$

$$\dot{\psi} + \psi \frac{\partial f}{\partial x} + e^{-\rho t} \frac{\partial g}{\partial x} = 0$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g \, ds$$

Transversality

$$\psi \rightarrow 0 \quad (t \rightarrow \infty)$$

Maximum flow

$$H \to \max$$

$$\dot{\psi} + \psi \frac{\partial f}{\partial x} + e^{-\rho t} \frac{\partial g}{\partial x} = 0$$

$$\dot{\psi} = -\psi \frac{\partial f}{\partial x} - e^{-\rho t} \frac{\partial g}{\partial x}$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g ds$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Maximum flow

$$u(t) : H \to \max$$

Adjoint equation

$$\dot{\psi} = -\psi \frac{\partial f}{\partial x} - e^{-\rho t} \frac{\partial g}{\partial x}$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g ds$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

$$p = e^{\rho t}\psi$$

Adjoint equation

$$\dot{\psi} = -\psi \frac{\partial f}{\partial x} - e^{-\rho t} \frac{\partial g}{\partial x}$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Maximum flow

$$u(t) : H \to \max$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g \, ds$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

$$p = e^{\rho t}\psi$$

Maximum flow

$$u(t) : H \to \max$$

Optimal flow

$$H = \rho \int_t^{\infty} e^{-\rho s} g ds$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

$$p = e^{\rho t} \psi$$

$$M = e^{\rho t} H = pf + g$$

Current value Hamiltonian

Maximum flow

$$u(t) : H \to \max$$

Optimal flow

$$H = \rho \int_t^\infty e^{-\rho s} g ds$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

$$p = e^{\rho t} \psi$$

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Current value Hamiltonian

Maximum flow

$$u(t) : H \to \max$$

$$u(t) : M \to \max$$

Optimal flow

$$H = \rho \int_t^\infty e^{-\rho s} g ds$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

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Current value Hamiltonian

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Maximum flow

$$u(t) : M \to \max$$

Optimal flow

$$H = \rho \int_{t}^{\infty} e^{-\rho s} g ds$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

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$$M = e^{\rho t} H = pf + g$$

Current value Hamiltonian

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Maximum flow

$$u(t) : M \to \max$$

Optimal flow

$$H = \rho \int_t^\infty e^{-\rho s} g ds$$

$$M = e^{\rho t} \rho \int_t^\infty e^{-\rho s} g ds$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

Maximum flow

$$u(t): M \to \max$$

$$M = e^{\rho t} H = pf + g$$

Optimal flow

$$M = e^{\rho t} \rho \int_t^{\infty} e^{-\rho s} g ds$$

$$M_0 = \rho \int_0^{\infty} e^{-\rho s} g ds = J$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

Maximum flow

$$u(t) : M \to \max$$

Optimal flow

$$M_0 = J$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

Maximum flow

$$u(t) : M \to \max$$

Optimal flow

$$M_0 = J$$

Transversality

$$\psi \to 0 \quad (t \to \infty)$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

Maximum flow

$$u(t) : M \to \max$$

Optimal flow

$$M_0 = J$$

Transversality

$$pe^{-\rho t} \to 0 \quad (t \to \infty)$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

State equation

$$\dot{x} = f$$

Maximum flow

$$u(t) : M \rightarrow \max$$

Optimal flow

$$M_0 = J$$

Transversality

$$pe^{-\rho t} \rightarrow 0 \quad (t \rightarrow \infty)$$

Adjoint equation

$$\dot{p} = \rho p - \frac{\partial f}{\partial x} p - \frac{\partial g}{\partial x}$$

State equation

$$\dot{x} = f$$

Hamiltonian system

Maximum flow

$$u(t) : M \to \max$$

Optimal flow

$$M_0 = J$$

Transversality

$$pe^{-\rho t} \to 0 \quad (t \to \infty)$$

Adjoint equation

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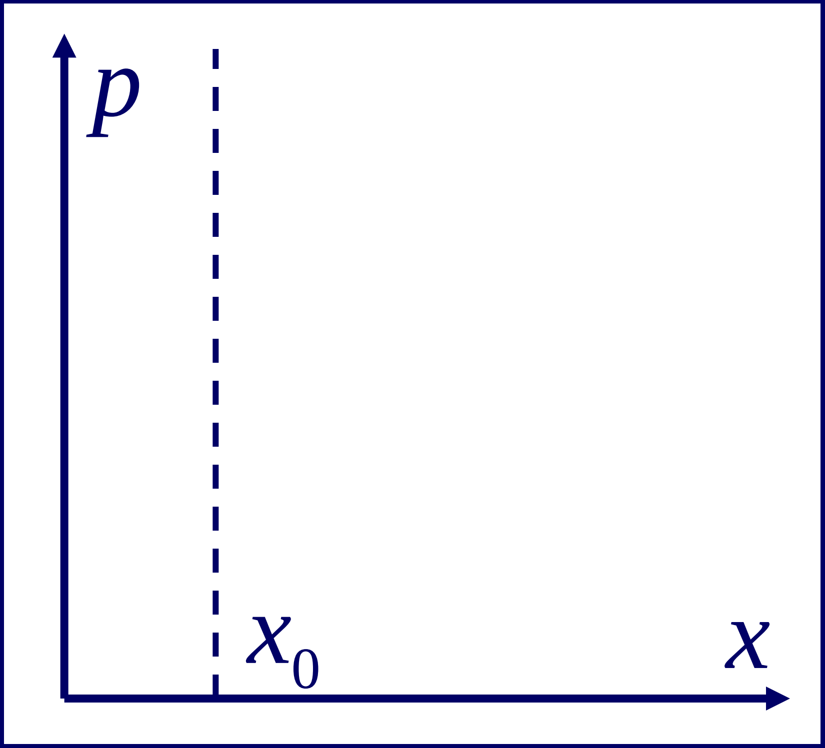

Transversality

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Adjoint equation

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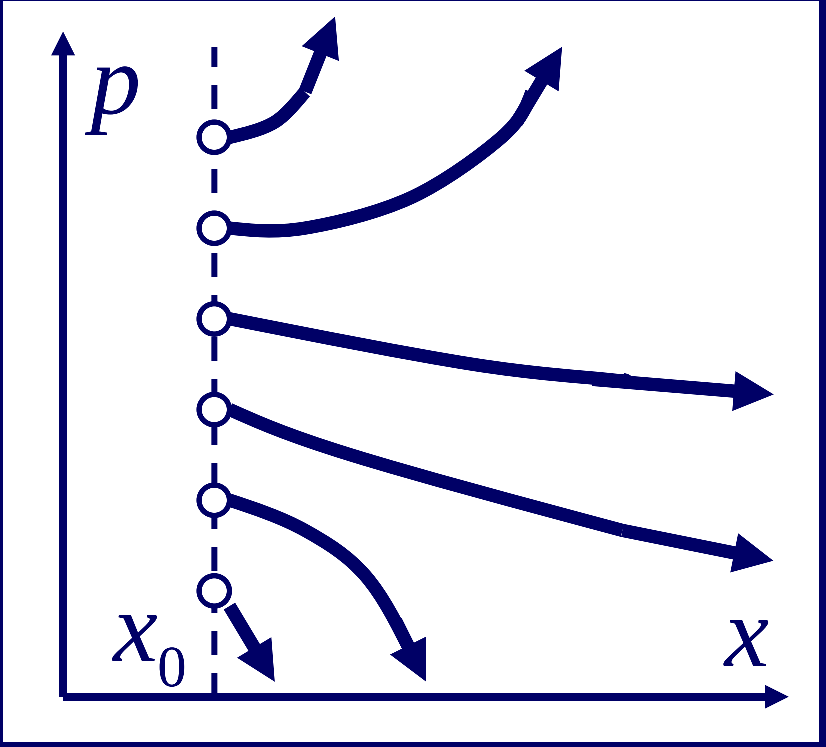

Adjoint equation

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Hamiltonian system

Transversality

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Maximum flow

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Optimal flow

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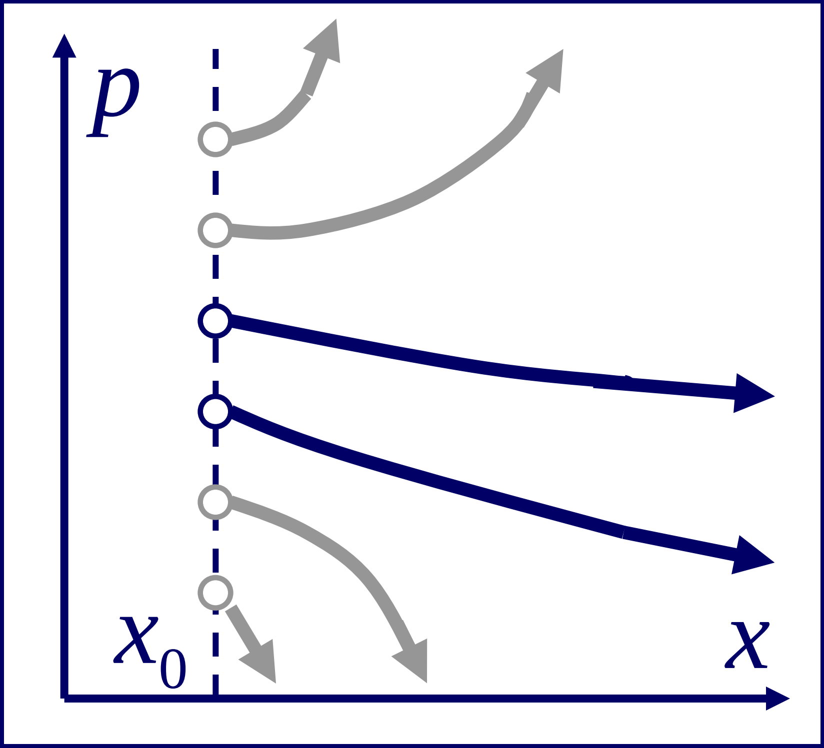

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Hamiltonian system

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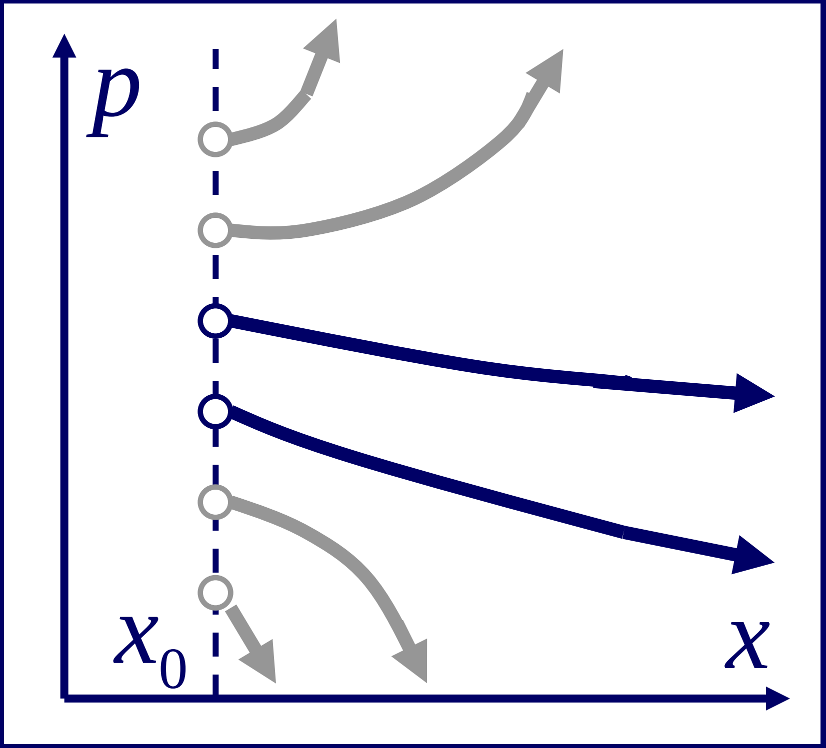

Adjoint equation

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Hamiltonian system

Transversality

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Optimal flow

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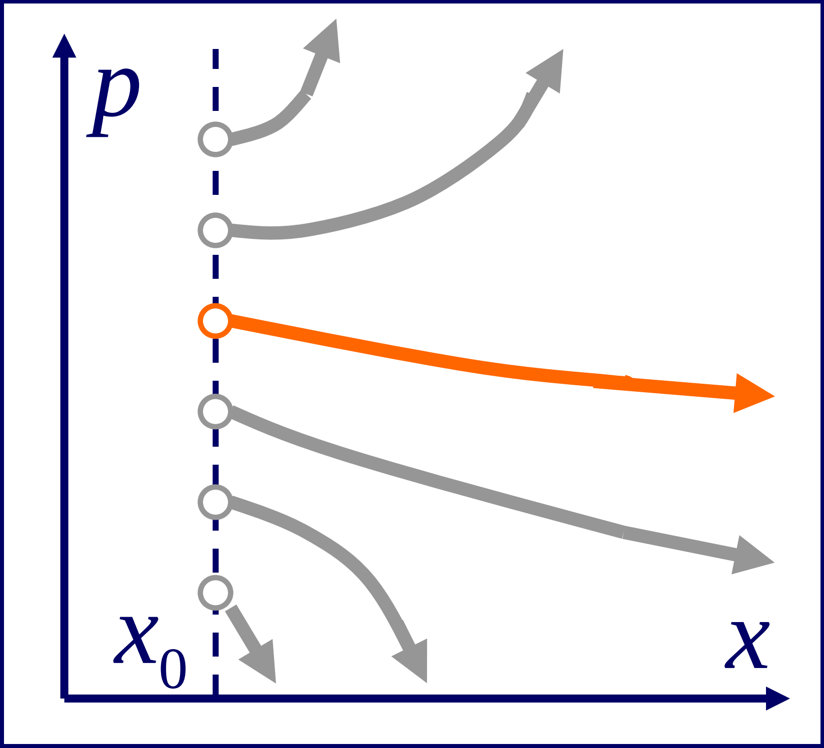

Thank you for your attention